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ABSTRACT

We are concerned with the hypothesis that two variables have a perfect disattenuated correlation, hence measure the same trait except for errors of measurement. This hypothesis is equivalent to saying, within the adopted model, that true scores of two psychological tests satisfy a linear relation. A statistical test of this hypothesis is derived when the relation is specified with the exception of the additive constant. Then the result is reinterpreted in terms of the possible existence of an unspecified linear relation between true scores of two psychological tests. A numerical example is provided by way of illustration. (Author)

TESTING WHETHER A DISATTENUATED CORRELATION IS PERFECT

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Let $X = T + E$ and $Y = U + F$ be two random variables made up of true scores, T and U, and errors of measurement, E and F. Suppose there is a linear relation between T and U. Then X and Y may be viewed as measuring the same dimension, but each individual measurement will be disturbed by an error. Therefore X and Y will be less than perfectly correlated.

The statistical problem of deciding whether a disattenuated correlation may be assumed to be perfect is of obvious practical significance. However, earlier techniques proposed for this purpose have been assessed by Lord (in press) as "cumbersome, approximate, or flawed." He suggested instead a procedure based on the construction of a confidence interval for the coefficients of a linear relation between true scores. This procedure is an adaptation of a result by Villegas (1964). However, a simple technique for testing whether a disattenuated correlation is perfect can be obtained even under much less restrictive assumptions.

The approach to be presented is a correlational one. Suppose that each test has been divided into two parts with observed scores X_1, X_2, and Y_1, Y_2, true scores T_1, T_2, and U_1, U_2, and errors E_1, E_2, and F_1, F_2. We have $T = T_1 + T_2$ and $U = U_1 + U_2$. Let the division of the tests be such that true scores on the parts of a given test may differ only by a constant. Errors will be assumed to be multinormally distributed and to satisfy the following conditions: (a) Variables E_1, E_2, F_1, F_2

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Abstract

We are concerned with the hypothesis that two variables have a perfect disattenuated correlation, hence measure the same trait except for errors of measurement. This hypothesis is equivalent to saying, within the adopted model, that true scores of two psychological tests satisfy a linear relation. A statistical test of this hypothesis is derived when the relation is specified with the exception of the additive constant. Then the result is reinterpreted in terms of the possible existence of an unspecified linear relation between true scores of two psychological tests. A numerical example is provided by way of illustration.

are pairwise independent with the possible exceptions σE_1F_1, $\sigma E_2F_2 \neq 0$.
(b) Variables T, U are independent of variables E_1, E_2, F_1, F_2.

The linear hypothesis we wish to test on the basis of a sample of observations is H_0: $\beta_1 T + \beta_2 U + \gamma = 0$ with specified coefficients $\beta_1 \neq 0$ and β_2. The additive constant γ remains unspecified.

Let us introduce the following new variables:

$$W_1 = \beta_1X_1 + \beta_2Y_1 = \beta_1T_1 + \beta_2U_1 + \beta_1E_1 + \beta_2F_1$$

$$W_2 = \beta_1X_2 + \beta_2Y_2 = \beta_1T_2 + \beta_2U_2 + \beta_1E_2 + \beta_2F_2 \,.$$

Under H_0 the sums $\beta_1T_1 + \beta_2U_1$ and $\beta_2T_2 + \beta_2U_2$ are constants. Variables W_1 and W_2 are independent precisely when H_0 is correct. But when H_0 is false, then W_1 and W_2 will be positively correlated, since

$$\sigma_{W_1W_2} = \frac{1}{4}\,\sigma^2_{\beta_1T+\beta_2U} \quad .$$

Under H_0 variables W_1 and W_2 need not have equal expectations or equal variances. The problem reduces to testing whether W_1 and W_2 may be uncorrelated when no restrictions are imposed on means and variances. A one-sided statistical test is required.

Let the sample variance-covariance matrix $\underset{\sim}{V}$ of the partitioned vector $(X_1,Y_1:X_2,Y_2)$ be partitioned accordingly with submatrices $\underset{\sim}{V}_{11}$, $\underset{\sim}{V}_{12}$, $\underset{\sim}{V}'_{12}$ and $\underset{\sim}{V}_{22}$. Matrices $\underset{\sim}{V}_{11}$, $\underset{\sim}{V}_{22}$ are positive definite. The sample correlation of W_1 and W_2 is

$$r = \underset{\sim}{\beta}'\underset{\sim}{V}_{12}\underset{\sim}{\beta}(\underset{\sim}{\beta}'\underset{\sim}{V}_{11}\underset{\sim}{\beta}\underset{\sim}{\beta}'\underset{\sim}{V}_{22}\underset{\sim}{\beta})^{-\frac{1}{2}}$$

when $\underset{\sim}{\beta}' = (\beta_1, \beta_2)$. Knowledge of this quantity enables us to carry out the test of H_0. We may also utilize it in constructing a "confidence interval" for $\underset{\sim}{\beta}$.

This result is applicable when the existence of a linear relation between true scores of two psychological tests is assumed. However, a linear relation may not exist at all. This is equivalent to saying that the disattenuated correlation between two psychological tests may be less than perfect. In fact, we may be primarily interested in testing whether there is <u>any</u> linear relation between true scores. The assumption of such an unspecified linear relation will be denoted by H_0^*. It is our aim now to give a statistical test of H_0^*.

If H_0^* is correct then there is (up to multiplication) precisely <u>one</u> nontrivial linear relation between true scores, $\beta_1 T + \beta_2 U + \gamma = 0$, with unknown values β_1, β_2, γ. Let us seek the minimum of r when $\underset{\sim}{\beta}$ varies over all nonzero vectors. This minimum may be compared with the critical value (level a, df = N - 2) of a product-moment correlation under normality assumptions when the population correlation is zero. The proposed technique will be conservative, i.e., when rejection of H_0^* occurs, then the true corresponding level a^* will be $a^* \leq a$. Let us write

$$r^* = \min_{\underset{\sim}{\beta} \neq 0} r.$$

We will think of $\underset{\sim}{V}_{12}$ as a symmetric matrix. Obviously, substitution of $(\underset{\sim}{V}_{12} + \underset{\sim}{V}'_{12})/2$ for $\underset{\sim}{V}_{12}$ is allowable. We will further suppose that V_{12} is positive definite. Otherwise r^* would not be a positive quantity and rejection of H_0^* would not be possible at any level a.

Quantity r^* may be determined as follows. Let $V_{12} = P\Delta P'$ with P orthogonal and Δ positive diagonal and define

$$\xi = \Delta^{\frac{1}{2}}P'\beta(\beta'P\Delta P'\beta)^{-\frac{1}{2}} ,$$

$$A = \Delta^{-\frac{1}{2}}P'V_{11}P\Delta^{-\frac{1}{2}} ,$$

$$B = \Delta^{-\frac{1}{2}}P'V_{22}P\Delta^{-\frac{1}{2}} .$$

We derive that

$$(r^*)^{-2} = \max_{|\xi|=1} \xi'A\xi\xi'B\xi \quad .$$

Writing $\xi' = (\cos\ \phi, \sin\ \phi)$ we seek maximization of the function $\eta(\phi) = \xi'A\xi\xi'B\xi$ under variation of ϕ. Setting $d\eta/d\phi = 0$ yields a quartic in ctg ϕ with coefficients that are simple functions of the elements of A and B. The quartic has at least two real solutions.

A numerical example will be appended by way of illustration. Lord (1957) reported the following observed variance-covariance matrix V for $N = 649$:

$$\begin{pmatrix} 86.40 & 57.78 & 56.87 & 58.30 \\ 57.78 & 86.26 & 59.32 & 59.67 \\ 56.87 & 59.32 & 97.29 & 73.82 \\ 58.90 & 59.67 & 73.82 & 97.82 \end{pmatrix}$$

The first two variables represent parallel halves of a vocabulary test administered under very liberal time limits. The last two variables represent parallel halves of a vocabulary test constructed so as to be as nearly equivalent as possible to the halves of the first test, except that the time of administration was so short that only 2% of the examinees completed each half. We wish to test the hypothesis that the two tests measure the same trait except for errors of measurement regardless of speed conditions.

It will be noted that parallelism of corresponding test halves has not at all been assumed in deriving our statistical technique. Hence the data of the present example satisfy stricter conditions than required.

The necessary submatrices $\underset{\sim}{V}_{11}$, $\underset{\sim}{V}_{22}$ and $\underset{\sim}{V}_{12}$ can be easily determined from $\underset{\sim}{V}$. Actually, we replace the original $\underset{\sim}{V}_{12}$ by $(\underset{\sim}{V}_{12} + \underset{\sim}{V}'_{12})/2$ and establish the canonical decomposition of this latter matrix. For the present example, the quartic in ctg ϕ was found to have two real roots. The minimal correlation r^* became .1839. It is now convenient to use the transformation

$$t^* = r^*[(N - 2)/(1 - r^{*2})]^{1/2}$$

which gives $t^* = 4.76$. This value is to be interpreted as a one-sided t with $df = 647$. We find that $a < .005$ and conclude that the trait captured by vocabulary tests depends on speed conditions.

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Footnote

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